

BAIRE THEOREM FOR IDEALS OF SETS

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ABSTRACT. We study ideals \mathcal{I} on \mathbb{N} satisfying the following Baire-type property: if X is a complete metric space and $\{X_A : A \in \mathcal{I}\}$ is a family of nowhere dense subsets of X with $X_A \subset X_B$ whenever $A \subset B$, then $\bigcup_{A \in \mathcal{I}} X_A \neq X$. We give several characterizations and determine the ideals having this property among certain classes like analytic ideals and P-ideals. We also discuss similar covering properties when considering families of compact and meager subsets of X .

1. INTRODUCTION

For a given set X we denote as usual by $\mathcal{P}(X)$ the collection of all subsets of X . We call a set $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ an *ideal* if $\mathbb{N} \notin \mathcal{I}$ and given $A, B \in \mathcal{I}$ we have that $\mathcal{P}(A) \subset \mathcal{I}$ and $A \cup B \in \mathcal{I}$. A set $\beta \subset \mathcal{I}$ is a *basis* of \mathcal{I} if every $A \in \mathcal{I}$ is contained in some $B \in \beta$. The *character* of \mathcal{I} is the minimal cardinality of a basis of \mathcal{I} . Along this paper, every considered ideal \mathcal{I} is supposed to contain the ideal Fin of all finite subsets of \mathbb{N} .

As a dual concept, a set $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ is a *filter* on \mathbb{N} if $\{\mathbb{N} \setminus A : A \in \mathcal{F}\}$ is an ideal on \mathbb{N} , and $\beta \subset \mathcal{F}$ is called a *basis* of \mathcal{F} if every $A \in \mathcal{F}$ contains some $B \in \beta$. If $(x_n)_{n \in \mathbb{N}}$ is a sequence in a topological space X , then it is said to be *\mathcal{F} -convergent* to $a \in X$, usually written $a = \lim_{n, \mathcal{F}} x_n$, if for every neighbourhood V of a we have that $\{n \in \mathbb{N} : x_n \in V\}$ belongs to \mathcal{F} .

Let \mathcal{F} be a filter on \mathbb{N} and let E be an arbitrary Banach space. A sequence $(e_n)_{n \in \mathbb{N}}$ in E is said to be an *\mathcal{F} -basis* of E if for every $x \in E$ there exists a unique sequence of scalars $(a_n)_{n \in \mathbb{N}}$ such that

$$x = \lim_{n, \mathcal{F}} \sum_{i=1}^n a_i e_i.$$

This definition extends the notion of Schauder basis, which corresponds to the case $\mathcal{F}_{cf} := \{A \subset \mathbb{N} : \mathbb{N} \setminus A \in \text{Fin}\}$, known as the Fréchet filter. The concept of \mathcal{F} -basis was introduced in [5], but previously considered in [2] for the filter of

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statistical convergence

$$\mathcal{F}_{st} := \left\{ A \subset \mathbb{N} : \lim_n \frac{|A \cap \{1, \dots, n\}|}{n} = 1 \right\}.$$

It is clear from the definition of \mathcal{F} -basis that the coefficient maps $e_n^*(x) = a_n$ are linear on E . However, and in contrast with Schauder bases, it is not known whether the e_n^* 's are necessarily continuous. A partial result was given by T. Kochanek [8], who showed that if \mathcal{F} has character less than \mathfrak{p} then the answer is positive. Here \mathfrak{p} denotes the *pseudointersection number*, defined as the minimum of the cardinals κ for which the following claim is true: *if \mathcal{A} is a family of subsets of \mathbb{N} with cardinality less than κ and satisfying that $\bigcap \mathcal{A}_0$ is infinite for each finite subset $\mathcal{A}_0 \subset \mathcal{A}$, then there is an infinite set $B \subset \mathbb{N}$ such that $B \setminus A$ is finite for every $A \in \mathcal{A}$.*

If we work with the dual ideal \mathcal{I} associated to \mathcal{F} , a review of Kochanek's argument shows that the key step to get the result is that \mathcal{I} has the next property:

- (\square) If X is a complete metric space and $\{X_A : A \in \mathcal{I}\}$ is a set of meager subsets of X with $X_A \subset X_B$ whenever $A \subset B$, then $\bigcup \{X_A : A \in \mathcal{I}\} \neq X$.

Unfortunately not every ideal has this property. If the character of \mathcal{I} is less than \mathfrak{p} , then it has property (\square), since $\bigcup \{X_A : A \in \mathcal{I}\}$ is equal to $\bigcup \{X_B : B \in \beta\}$ which is the union of less than \mathfrak{p} meager subsets, and this is again a meager subset of X by [4, Corollary 22C]. In section 2, we show that the converse is also true under the set-theoretical assumption $\mathfrak{p} = \mathfrak{c}$.

The aim of this paper is to study what happens if we replace the condition “meager” by “nowhere dense” in (\square). Ideals satisfying this last property will be called *Baire ideals*. In section 3 we prove several characterizations of this type of ideals. We also show that in order to demonstrate that an ideal \mathcal{I} is a Baire ideal, we just have to check property (\square) (with nowhere dense subsets instead of meager ones) for the metrizable space $X = D^{\mathbb{N}}$, D being the discrete space of cardinality equal to \mathfrak{c} .

The fourth section is devoted to determine which are the Baire ideals in the classes of analytic ideals and P-ideals. Here we work in ZFC without any other set-theoretical assumptions. We show that in both cases the only Baire ideals are the countably generated ideals. In contrast with this, we construct in section 5 a model of ZFC in which we can find an uncountably generated P-ideal \mathcal{I} satisfying property (\square) for the particular case $X = 2^{\mathbb{N}}$. We also study the case of ideals generated by an almost disjoint family of subsets of \mathbb{N} .

In the last section, we show that if in (\square) one considers compact subsets instead of meager ones, then there are uncountably generated F_σ ideals satisfying that property for $X = \mathbb{N}^{\mathbb{N}}$.

Our notation and terminology is standard and it is either explained when needed or can be found in [6] and [7].

2. BAIRE IDEALS

As we announced in the introduction, if we assume that $\mathfrak{p} = \mathfrak{c}$ (for instance, under Martin's Axiom), property (\square) depends exclusively on the character of the ideal \mathcal{I} , as the following proposition shows.

Proposition 2.1. *If \mathcal{I} has a basis of cardinality less than \mathfrak{p} then it has property (\square) . If $\mathfrak{p} = \mathfrak{c}$, then the converse is true.*

Proof. The first part was discussed in the introduction so we just prove the second statement. Suppose that \mathcal{I} is an ideal without a basis of cardinality less than \mathfrak{p} . Then we can easily construct a basis $\{B_\alpha : \alpha \in \mathfrak{p}\}$ with the property that B_γ does not belong to the ideal generated by $\{B_\alpha : \alpha < \gamma\}$. Consider the map $g(\cdot) : \mathcal{I} \rightarrow \mathfrak{p}$ that associates to each $A \in \mathcal{I}$ the minimum of the ordinals γ such that A belongs to the ideal generated by $\{B_\alpha : \alpha < \gamma\}$. Now we can enumerate the elements of \mathbb{R} as $\{x_\alpha : \alpha \in \mathfrak{p}\}$ and define for each $A \in \mathcal{I}$ the set $\mathbb{R}_A := \{x_\alpha : \alpha \leq g(A)\}$. This is a meager set (union of less than \mathfrak{p} meager sets is meager by [4, Corollary 22C]), and obviously $\mathbb{R}_A \subset \mathbb{R}_B$ whenever $A \subset B$. But it is also clear that $\bigcup \{\mathbb{R}_A : A \in \mathcal{I}\} = \mathbb{R}$ since $x_\alpha \in \mathbb{R}_{B_\alpha}$ for every $\alpha < \mathfrak{p}$. \square

It is natural to ask whether the situation is the same if we restrict to nowhere dense subsets instead of meager ones. Given a topological space X , we will denote by $\text{NWD}(X)$ the family of all nowhere dense subsets of X ordered with inclusion.

Definition 2.2. *Let \mathcal{I} be an ideal on \mathbb{N} and X a topological space. We call \mathcal{I} a Baire ideal for X if for every monotonic function $f : \mathcal{I} \rightarrow \text{NWD}(X)$ we have that*

$$\bigcup \{f(A) : A \in \mathcal{I}\} \neq X.$$

If \mathcal{I} has this property for every complete metric space X then we simply say that \mathcal{I} is a Baire ideal.

It follows from Proposition 2.1 that every ideal \mathcal{I} with character less than \mathfrak{p} is a Baire ideal. However, it is not clear if the converse is true, even under Martin's Axiom. It is interesting to remark that if we put less restrictions on X then we can give a full characterization.

Proposition 2.3. *Let \mathcal{I} be an ideal on \mathbb{N} . Then, \mathcal{I} is countably generated if and only if it is a Baire ideal for every topological space X of second category in itself.*

Proof. If X is of second category in itself then it is not the union of a countable family of nowhere dense subsets. Hence, by the comments preceding this Proposition, if \mathcal{I} is countably generated then it is a Baire ideal for X .

To see the converse, suppose that \mathcal{I} is not countably generated. We can endow $X = \mathcal{I}$ with the topology τ generated by the basis of closed sets consisting of \emptyset and the sets

$$X_A = \{I \in \mathcal{I} : I \subset A\} \quad (A \in \mathcal{I}).$$

Every nowhere dense subset of (X, τ) is contained in some X_A , and moreover every X_A is nowhere dense since if a basic open $X \setminus X_B$ was contained in X_A , then we would have that $X = X_A \cup X_B$, contradicting that \mathcal{I} is not countably generated. The same fact shows that $X \neq \bigcup_{n \in \mathbb{N}} X_{A_n}$ for every sequence $(A_n)_{n \in \mathbb{N}}$ in X , so X is not the union of a countable family of nowhere dense subsets and hence it is of second category in itself. Finally it is clear that $\bigcup \{X_A : A \in \mathcal{I}\} = X$ and so \mathcal{I} is not a Baire ideal for X . \square

3. CHARACTERIZATIONS OF BAIRE IDEALS

Recall that the *weight* $w(X)$ of a topological space is the minimum cardinal κ for which X has basis of open sets with such cardinality. In the current section, we will show that in order to prove that \mathcal{I} is a Baire ideal for every complete metric space X of weight $w(X) \leq \kappa$ we just have to check that it is a Baire ideal for $D^{\mathbb{N}}$ where D is a discrete space of cardinality κ . Moreover, we give another characterization of this fact in terms of the properties of certain subtrees of $\mathcal{P}(\mathcal{I})$.

If D is a discrete space then $D^{\mathbb{N}}$ with the product topology is completely metrizable. We introduce some notation: If $\sigma : \mathbb{N} \rightarrow D$ is an element of $D^{\mathbb{N}}$ then we put $\sigma|_0 := \emptyset$ and $\sigma|_k = (\sigma(1), \dots, \sigma(k))$ if $k > 0$. We denote by $D^{<\mathbb{N}}$ the set of finite sequences of elements in D . In other words, $t \in D^{<\mathbb{N}}$ if there is $\sigma \in D^{\mathbb{N}}$ and $k \geq 0$ such that $\sigma|_k = t$. In this case we will write $t \preceq \sigma$. If $s \in D^{<\mathbb{N}}$ we write $s \preceq t$ if t is an extension of s ; i.e. if there are $\sigma \in D^{\mathbb{N}}$ and $0 \leq p \leq q$ with $s = \sigma|_p$, $t = \sigma|_q$. If $\alpha \in D$ then we denote by $s^\wedge \alpha := (s(1), \dots, s(k), \alpha)$ the extended sequence obtained by adding the element α at the end of s . With this notation, a basis of open subsets in $D^{\mathbb{N}}$ is given by the sets

$$U_s = \{\sigma \in D^{\mathbb{N}} : \sigma \succ s\} \quad (s \in D^{<\mathbb{N}}).$$

Notice that each U_s is clopen. Recall that $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$ is called *hereditary* if $B \subset A \in \mathcal{A}$ implies that $B \in \mathcal{A}$.

Theorem 3.1. *Let D be a discrete space of (infinite) cardinality κ and \mathcal{I} an ideal on \mathbb{N} . The following assertions are equivalent:*

- (1) \mathcal{I} is a Baire ideal for every complete metric space X with $w(X) \leq \kappa$.
- (2) \mathcal{I} is $D^{\mathbb{N}}$ -Baire.
- (3) Let $f : D^{<\mathbb{N}} \rightarrow \mathcal{P}(\mathcal{I})$ be monotonic and such that for every $s \in D^{<\mathbb{N}}$:
 (i) $f(s)$ is hereditary, (ii) $\mathcal{I} = \bigcup_{t \succ s} f(t)$.
 Then, there exists $\sigma \in D^{\mathbb{N}}$ such that $\bigcup_{k \in \mathbb{N}} f(\sigma|_k) = \mathcal{I}$.
- (4) Let $f : D^{<\mathbb{N}} \rightarrow \mathcal{P}(\mathcal{I})$ be monotonic and such that for every $s \in D^{<\mathbb{N}}$:
 (i) $f(s)$ is hereditary, (ii') $\mathcal{I} = \bigcup_{d \in D} f(s^\wedge d)$, (iii) $f(s) = \bigcap_{d \in D} f(s^\wedge d)$.
 Then, there exists $\sigma \in D^{\mathbb{N}}$ such that $\bigcup_{k \in \mathbb{N}} f(\sigma|_k) = \mathcal{I}$.

Proof. Implications (1) \Rightarrow (2) and (3) \Rightarrow (4) are obvious.

(2) \Rightarrow (3): Suppose that we have a function $f : D^{<\mathbb{N}} \rightarrow \mathcal{P}(\mathcal{I})$ as in (3) and define the map $F : \mathcal{I} \rightarrow \text{NWD}(D^{\mathbb{N}})$ that assigns to each $A \in \mathcal{I}$ the set

$$F(A) = \{\sigma \in D^{\mathbb{N}} : A \not\subseteq f(\sigma|_k) \text{ for every } k \in \mathbb{N}\}.$$

To see that $F(A)$ is effectively nowhere dense, notice that if U_s is a basic open then there exists by (ii) an element $t \succ s$ with $A \in f(t)$, which means that $U_t \subset U_s \setminus F(A)$. Since we are assuming that \mathcal{I} is a Baire ideal for $D^{\mathbb{N}}$, we can find $\sigma \in D^{\mathbb{N}} \setminus \bigcup \{F(A) : A \in \mathcal{I}\}$. But $\sigma \notin F(A)$ means that $A \in \bigcup_{k \in \mathbb{N}} f(\sigma|_k)$ for every $A \in \mathcal{I}$.

(4) \Rightarrow (1): Let $\{W_d : d \in D\}$ be a basis of open sets in X (repeating elements if necessary). We are going to construct a collection $\{V_s : s \in D^{<\mathbb{N}}\}$ of open sets in the following inductive way: $V_\emptyset = X$ and $V_{(d)} := W_d$ for each $d \in D$. Suppose that we have constructed V_s ($s \in D^{<\mathbb{N}}$). The elements $V_{s^\wedge x}$ ($x \in D$) are going to be all the open sets W_d such that $\overline{W_d} \subset V_s$ and $\text{diam}(W_d) \leq \text{diam}(V_s)/2$ repeating elements if necessary (here $\text{diam}(C)$ means the diameter of the set C). Observe that for every $\sigma \in D^{\mathbb{N}}$, the branch $\{V_{\sigma|_k} : k \in \mathbb{N}\}$ has non-empty intersection which consists of a unique element that we will denote by p_σ . Of course different branches could determine the same point, and it is also clear that every element of X belongs to the intersection of one of these branches. Moreover, each open set V_s can be described now as $V_s = \{p_\sigma : \sigma \succ s\}$.

Given $F : \mathcal{I} \rightarrow \text{NWD}(X)$ monotonic, define a new map $f : D^{<\mathbb{N}} \rightarrow \mathcal{P}(\mathcal{I})$ as

$$f(s) = \{A \in \mathcal{I} : \overline{F(A)} \cap V_s = \emptyset\} \text{ for each } s \in D^{<\mathbb{N}}.$$

It is also monotonic and satisfies condition (i). To see (ii'), fix $s \in D^{<\mathbb{N}}$ and take an arbitrary $A \in \mathcal{I}$. Since $F(A)$ is nowhere dense, there exists an open set W_d such that $\overline{W_d} \subset V_s \setminus F(A)$ and $\text{diam}(W_d) \leq \text{diam}(V_s)/2$. We know that this W_d is equal to some element $V_{s^\wedge x}$, so $A \in f(s^\wedge x)$. We check now condition (iii) on f . If $s \in D^{<\mathbb{N}}$ and $A \notin f(s)$ then $F(A) \cap V_s \neq \emptyset$, so there exists an element p_σ ($\sigma \succ s$) that belongs to this intersection. If $\sigma|_k = s$, then notice that $V_{\sigma|_{k+1}} = V_{s^\wedge \sigma(k+1)}$ also has non-empty intersection with $F(A)$, so $A \notin f(\sigma|_{k+1})$. This shows that $f(s) \supset \bigcap_{d \in D} f(s^\wedge d)$, but this is indeed an equality by monotonicity.

Using the assumption, there is $\sigma \in D^{\mathbb{N}}$ such that every $A \in \mathcal{I}$ belongs to $f(\sigma|_k)$ for some $k \in \mathbb{N}$, and so $F(A) \cap U_{\sigma|_k} = \emptyset$. This implies that $p_\sigma \notin F(A)$ for every $A \in \mathcal{I}$. \square

In the case of separable complete metric spaces (*Polish spaces*) we can give (apparently) simpler equivalent conditions than those of Theorem 3.1.

Corollary 3.2. *Let \mathcal{I} be an ideal on \mathbb{N} . The following assertions are equivalent:*

- (1) \mathcal{I} is a Baire ideal for every Polish space.
- (2') \mathcal{I} is a Baire ideal for $2^{\mathbb{N}}$.
- (3') let $f : 2^{<\mathbb{N}} \rightarrow \mathcal{P}(\mathcal{I})$ be monotonic and such that for every $s \in 2^{<\mathbb{N}}$:
 - (i) $f(s)$ is hereditary, (ii) $\bigcup_{t \succ s} f(t) = \mathcal{I}$.
 Then, there is $\sigma \in 2^{\mathbb{N}}$ such that $\bigcup_{k \in \mathbb{N}} f(\sigma|_k) = \mathcal{I}$.

Proof. (1) \Rightarrow (2') is clear and (2') \Rightarrow (3') follows as in the proof of (2) \Rightarrow (3) in Theorem 3.1.

(3') \Rightarrow (1): We will show that (3) of Theorem 3.1 is satisfied for the discrete topological space $D = \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $f : \mathbb{N}_0^{<\mathbb{N}} \rightarrow \mathcal{P}(\mathcal{I})$ be a monotonic function such that for every $s \in \mathbb{N}_0^{<\mathbb{N}}$ we have that $f(s)$ is hereditary and $\bigcup_{t \succ s} f(t) = \mathcal{I}$.

We establish now a monotonic onto map $\psi: 2^{<\mathbb{N}} \rightarrow \mathbb{N}_0^{<\mathbb{N}}$. Each $s \in 2^{<\mathbb{N}}$ can be seen as a sequence of blocks of 0's separated by 1, so it can be written as $s = (0^{n_1}, 1, 0^{n_2}, 1, \dots, 1, 0^{n_{k+1}})$ where $0^0 := \emptyset$ and $0^k := (0, \dots, 0)$ (k times) if $k > 0$. Put $\psi(s) = (n_1, \dots, n_k)$ with the convention $\psi((0^k)) = \emptyset$ for each $k \geq 0$. It satisfies the conditions above, so the map $g: 2^{<\mathbb{N}} \rightarrow \mathcal{P}(\mathcal{I})$ given by $g(s) = f(\psi(s))$ is monotonic and clearly satisfies (i) and (ii). By hypothesis there is $\sigma \in 2^{\mathbb{N}}$ such that $\bigcup_{k \in \mathbb{N}} g(\sigma|_k) = \mathcal{I}$. There are two possibilities: If there is $k_0 \in \mathbb{N}$ such that $\sigma(k) = 0$ whenever $k \geq k_0$ then $\psi(\sigma|_k) = \psi(\sigma|_{k_0})$ for $k \geq k_0$ and we conclude that $g(\sigma|_{k_0}) = \mathcal{I}$. On the other hand, if the support of σ is not finite, we can write $\sigma = (0^{n_1}, 1, 0^{n_2}, 1, \dots)$ so $\tau = (n_1, \dots, n_k, \dots) \in \mathbb{N}_0^{\mathbb{N}}$ satisfies

$$\bigcup_{k \in \mathbb{N}} f(\tau|_k) = \bigcup_{k \in \mathbb{N}} g((0^{n_1}, 1, 0^{n_2}, 1, \dots, 0^{n_k}, 1)) = \mathcal{I}.$$

□

Remark 3.3. Since $2^{\mathbb{N}}$ is a compact space, the statements (1), (2') and (3') of Corollary 3.2 are also equivalent to “ \mathcal{I} is a Baire ideal for every compact metric space”. Recall that the dual filters of these ideals were introduced in [1, Definition 4.2] under the name of category respecting filters, as an example of filters having the Schur's property: every weakly \mathcal{F} -convergent to 0 sequence in ℓ^1 is \mathcal{F} -convergent in norm to zero.

Looking at Theorem 3.1 it is natural to ask whether there is a complete metric space X_0 such that \mathcal{I} is a Baire ideal for every complete metric space X (independently of the weight of the space) whenever it is a Baire ideal for X_0 . The last part of this section is devoted to find such a space.

Proposition 3.4. Let X be a complete metric space with $|X| > \mathfrak{c}$ and $\{D_\alpha\}_{\alpha \in \mathfrak{c}}$ a family of non-empty closed elements of $\text{NWD}(X)$. Then, there exists a closed subset $\Omega \subseteq X$ with $|\Omega| \leq \mathfrak{c}$ and such that $D_\alpha \cap \Omega$ is non-empty and belongs to $\text{NWD}(\Omega)$.

Proof. We start by constructing an increasing sequence $(\Omega_n)_{n \geq 0}$ of subsets of X in the following way. Take Ω_0 an arbitrary set with $|\Omega_0| \leq \mathfrak{c}$ and $\Omega_0 \cap D_\alpha \neq \emptyset$ for every $\alpha \in \mathfrak{c}$. Now suppose that Ω_n has been constructed. Using that each D_α is nowhere dense in X , for each $\alpha \in \mathfrak{c}$ and every $p \in \Omega_n \cap D_\alpha$ we can find a sequence $(y_k^{p,\alpha})_{k \in \mathbb{N}}$ in $X \setminus D_\alpha$ which converges to p . Then define

$$\Omega_{n+1} := \Omega_n \cup \{y_k^{p,\alpha} : p \in D_\alpha \cap \Omega_n, \alpha \in \mathfrak{c}, k \in \mathbb{N}\}.$$

Define Ω as the closure of $\bigcup_{n \in \mathbb{N}} \Omega_n$ in X . Notice that Ω has cardinality at most the continuum since each of its elements is the limit of a sequence in $\bigcup_{n \in \mathbb{N}} \Omega_n$, which has cardinality less or equal than \mathfrak{c} . To finish the proof we have to check that $D_\alpha \cap \Omega \in \text{NWD}(\Omega)$ for every $\alpha \in \mathfrak{c}$. Fix such an α and let U be an open subset of X with $U \cap \Omega \neq \emptyset$. We can find $n_0 \in \mathbb{N}$ with $U \cap \Omega_{n_0} \neq \emptyset$, and by construction there exists $y_k^{p,\alpha} \in U \cap \Omega_{n_0+1} \setminus D_\alpha$, so $U \cap \Omega$ cannot be contained in $D_\alpha \cap \Omega$. □

Corollary 3.5. *Let \mathcal{I} be an ideal on \mathbb{N} . Then, \mathcal{I} is a Baire ideal (for every complete metric space) whenever it is a Baire ideal for $D^{\mathbb{N}}$ where D is a discrete space with $|D| = \mathfrak{c}$.*

Proof. Let X be a complete metric space. If $|X| \leq \mathfrak{c}$ then \mathcal{I} is a Baire ideal for X by Theorem 3.1. We prove now that this is also true if X has cardinality strictly bigger than the continuum. Given a monotonic function $F: \mathcal{I} \rightarrow \text{NWD}(X)$ we can assume that its images are closed sets by taking the closure. Proposition 3.4 provides a closed subset Ω of X of cardinality at most \mathfrak{c} such that the map $G: \mathcal{I} \rightarrow \text{NWD}(\Omega)$ given by $G(A) = F(A) \cap \Omega$ is well-defined and monotonic. Therefore $\Omega \neq \bigcup \{G(A): A \in \mathcal{I}\}$ since \mathcal{I} is a Baire ideal for Ω , which implies that $X \neq \bigcup \{F(A): A \in \mathcal{I}\}$. \square

4. BAIRE IDEALS IN SOME CLASSES OF IDEALS

4.1. Analytic ideals. The ideals we are dealing with in this section are the mostly used one (see [3], [11]). Roughly speaking, all the ideals that can be defined by explicit formulas are analytic. Due to one of equivalent definitions, see [6, Theorem 7.9, Definition 14.1], a subset A of a Polish space X is *analytic* if it can be represented as an image of a continuous map acting from $\mathbb{N}^{\mathbb{N}}$ to X . In particular, all Borel sets are analytic. An ideal \mathcal{I} is *analytic* if it forms an analytic subset of the Polish topological space $2^{\mathbb{N}}$.

Proposition 4.1. *If \mathcal{I} is an analytic ideal on \mathbb{N} which is not countably generated, then it is not a Baire ideal for some Polish space.*

Proof. Suppose that there exists a continuous function $g: \mathbb{N}^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ whose image is the ideal \mathcal{I} . Following the notation of section 3, for each $s \in \mathbb{N}^{<\mathbb{N}}$ we write

$$U_s = \{\sigma \in \mathbb{N}^{\mathbb{N}}: \sigma \succ s\} \quad \text{and} \quad [s]^g := \{g(\sigma): \sigma \in \mathbb{N}^{\mathbb{N}}, \sigma \succ s\}.$$

We consider now the following set

$$X := \mathbb{N}^{\mathbb{N}} \setminus \bigcup \{U_s: [s]^g \text{ is contained in a countably generated subideal of } \mathcal{I}\}.$$

It is non-empty since $[\emptyset]^g = \mathcal{I}$ is not countably generated. Moreover, it is a closed subset of $\mathbb{N}^{\mathbb{N}}$ since the sets U_s are open, so X is a Polish space. Consider the map $F: \mathcal{I} \rightarrow \text{NWD}(X)$ defined for each $A \in \mathcal{I}$ as

$$F(A) = \{\sigma \in X: g(\sigma) \subset A\}.$$

Let us see that for every $A \in \mathcal{I}$ we have that $F(A) \in \text{NWD}(X)$. Let $U_s \cap X$ be a (relatively) open set in X . If $U_s \cap X \neq \emptyset$ then $[s]^g$ is not countably generated, which in particular implies that there is $\sigma \in \mathbb{N}^{\mathbb{N}}, \sigma \succ s$ such that $g(\sigma) \not\subset A$. Therefore there is $m \in \mathbb{N}$ with $g(\sigma) \cap \{1, \dots, m\} \not\subset A$ and by continuity we can find $\sigma \succ t \succ s$ with $g(\tau) \cap \{1, \dots, m\} = g(\sigma) \cap \{1, \dots, m\}$ for every $\tau \succ t$, which means that $\sigma \in U_t \subset U_s \setminus F(A)$. Hence, F is a well-defined monotonic function. On the other hand, $X = \bigcup \{F(A): A \in \mathcal{I}\}$ since $\sigma \in F(g(\sigma))$ for every $\sigma \in X$. \square

4.2. Generalized uncountably generated P-ideals. An ideal \mathcal{I} on \mathbb{N} is said to be a *P-ideal* if for every sequence $(A_n)_{n \in \mathbb{N}}$ in \mathcal{I} there exists $B \in \mathcal{I}$ such that $A_n \setminus B \in \text{Fin}$ for each $n \in \mathbb{N}$. Moreover, such an ideal \mathcal{I} is not countably generated if and only if for every $A \in \mathcal{I}$ there exists $B \in \mathcal{I}$ satisfying $B \setminus A \notin \text{Fin}$. This motivates the following definition.

Definition 4.2. Let $\mathcal{I}, \mathcal{I}_0$ be ideals on \mathbb{N} . We say that \mathcal{I} is a $P(\mathcal{I}_0)$ -ideal if it satisfies the following conditions:

- (I) Given $(A_n)_{n \in \mathbb{N}}$ in \mathcal{I} there is $A \in \mathcal{I}$ such that $A_n \setminus A \in \mathcal{I}_0$ for each $n \in \mathbb{N}$.
- (II) For each $A \in \mathcal{I}$ there exists $B \in \mathcal{I}$ such that $B \setminus A \notin \mathcal{I}_0$.

We show now how to construct $P(\mathcal{I}_0)$ -ideals which are not $P(\text{Fin})$ -ideals.

Example 4.3. Let \mathcal{I}, \mathcal{J} be ideals on \mathbb{N} where \mathcal{I} is an uncountably generated P-ideal. Take the direct sum $\mathcal{I} \oplus \mathcal{J}$, i.e. the family of all subsets $A \subseteq \mathbb{N} \times \{0, 1\} \cong \mathbb{N}$ such that

$$A^0 = \{n \in \mathbb{N} : (n, 0) \in A\} \in \mathcal{I} \quad \text{and} \quad A^1 = \{n \in \mathbb{N} : (n, 1) \in A\} \in \mathcal{J},$$

and write $\mathcal{I}' := \mathcal{I} \oplus \text{Fin}$, $\mathcal{J}' := \text{Fin} \oplus \mathcal{J}$. We claim that $\mathcal{I} \oplus \mathcal{J}$ is a $P(\mathcal{J}')$ -ideal. To see (I), let $(A_n)_{n \in \mathbb{N}}$ be a sequence of elements in $\mathcal{I} \oplus \mathcal{J}$. Since \mathcal{I} is a P-ideal, there exists $B \in \mathcal{I}$ such that $A_n^0 \setminus B \in \text{Fin}$ for every $n \in \mathbb{N}$, so

$$A_n \setminus (B \times \{0\}) \subseteq ((A_n^0 \setminus B) \times \{0\}) \cup (A_n^1 \times \{1\}) \in \mathcal{J}'.$$

Since \mathcal{I} is not countably generated, given $A \in \mathcal{I} \oplus \mathcal{J}$ we can find $B \in \mathcal{I}$ satisfying $B \setminus A^0 \notin \text{Fin}$, so $(B \times \{0\}) \setminus A \notin \mathcal{J}'$. This proves (II).

Observe that if $\mathcal{I} \oplus \mathcal{J}$ is a $P(\text{Fin})$ -ideal then \mathcal{J} is a P-ideal, since given a family $(J_n)_{n \in \mathbb{N}}$ in \mathcal{J} there exists $A \in \mathcal{I} \oplus \mathcal{J}$ such that $(J_n \times \{1\}) \setminus A \in \text{Fin}$ and hence $J_n \setminus A^1 \in \text{Fin}$ for every $n \in \mathbb{N}$. In particular taking \mathcal{J} not being a P-ideal we get that $\mathcal{I} \oplus \mathcal{J}$ is a $P(\mathcal{J}')$ -ideal but not a $P(\text{Fin})$ -ideal.

Recall that $(\ell^\infty, \|\cdot\|_\infty)$ is the Banach space of all bounded real sequences. If \mathcal{I}_0 is an ideal on \mathbb{N} , we write

$$c_0(\mathcal{I}_0) = \{(x_n)_{n \in \mathbb{N}} \in \ell^\infty : \{n \in \mathbb{N} : |x_n| > \varepsilon\} \in \mathcal{I}_0 \text{ for every } \varepsilon > 0\}.$$

It is easy to check that it is a closed subspace of ℓ^∞ . If $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ are sequences of real numbers, we will write $(x_n)_{n \in \mathbb{N}} \leq (y_n)_{n \in \mathbb{N}}$ if $x_n \leq y_n$ for each $n \in \mathbb{N}$. If $A \subset \mathbb{N}$, we will denote by χ_A the characteristic function of A .

Proposition 4.4. Let $\mathcal{I}, \mathcal{I}_0$ be ideals on \mathbb{N} such that \mathcal{I} is a $P(\mathcal{I}_0)$ -ideal. Then \mathcal{I} is not a Baire ideal.

Proof. We will find a closed subspace X of $E := \ell^\infty / c_0(\mathcal{I}_0)$ for which \mathcal{I} is not a Baire ideal. For each $A \in \mathcal{I}$ consider $F_A = \{[(x_n)_{n \in \mathbb{N}}] : 0 \leq (x_n)_{n \in \mathbb{N}} \leq \chi_A\}$. Notice that $[(x_n)_{n \in \mathbb{N}}] \in F_A$ if and only if there is $(y_n)_{n \in \mathbb{N}} \in c_0(\mathcal{I}_0)$ such that $0 \leq (x_n + y_n)_{n \in \mathbb{N}} \leq \chi_A$, or equivalently, if for every $\varepsilon > 0$ we have that

$$\{n \in \mathbb{N} \setminus A : |x_n| > \varepsilon\} \cup \{n \in A : x_n > 1 + \varepsilon\} \cup \{n \in A : x_n < -\varepsilon\} \in \mathcal{I}_0.$$

From this it easily follows that each F_A is closed. We can also deduce:

- (1) If $A, B \in \mathcal{I}$ then $A \setminus B \in \mathcal{I}_0$ if and only if $F_A \subseteq F_B$.
 Suppose that $A \setminus B \in \mathcal{I}_0$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence with $0 \leq (x_n)_{n \in \mathbb{N}} \leq \chi_A$. Define $(y_n)_{n \in \mathbb{N}}$ as $y_n = 0$ if $n \notin A \setminus B$ and $y_n = -x_n$ if $n \in A \setminus B$. It is clear that $(y_n)_{n \in \mathbb{N}} \in c_0(\mathcal{I}_0)$ and that $[(x_n)_{n \in \mathbb{N}}] = [(x_n)_{n \in \mathbb{N}} + (y_n)_{n \in \mathbb{N}}] \in F_B$. Conversely, if $F_A \subseteq F_B$ then $[\chi_A] \in F_B$ and the condition above implies that $A \setminus B \subset \{n \in \mathbb{N} \setminus B : \chi_A(n) > 1/2\} \in \mathcal{I}_0$.
- (2) For every $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{I}$ there exists $B \in \mathcal{I}$ such that $\bigcup_{n \in \mathbb{N}} F_{A_n} \subset F_B$.
 This follows from Definition 4.2 (I) and the previous property.
- (3) If $A \in \mathcal{I}$ there is $B \in \mathcal{I}$ such that F_A is a nowhere dense subset of F_B .
 By Definition 4.2 (II), there exists $B \in \mathcal{I}$ such that $B \setminus A \notin \mathcal{I}_0$. Replacing B by $A \cup B$, we can assume that $A \subset B$ so that $F_A \subset F_B$. To see that F_A is nowhere dense in F_B , take a sequence $0 \leq (x_n)_{n \in \mathbb{N}} \leq \chi_A$ and V an open set in E containing $[(x_n)_{n \in \mathbb{N}}]$. Fixed $0 < \delta < 1$, the sequence $(y_n)_{n \in \mathbb{N}} = (x_n)_{n \in \mathbb{N}} + \delta \chi_{B \setminus A}$ satisfies that $\|[(x_n)_{n \in \mathbb{N}}] - [(y_n)_{n \in \mathbb{N}}]\| \leq \delta$. We can take δ small enough so that $[(y_n)_{n \in \mathbb{N}}] \in V$. On the other hand, $[(y_n)_{n \in \mathbb{N}}] \notin F_A$ since $B \setminus A \subset \{n \in \mathbb{N} \setminus A : |y_n| > \delta/2\}$, but $[(y_n)_{n \in \mathbb{N}}] \in F_B$ because $0 \leq (y_n)_{n \in \mathbb{N}} \leq \chi_B$. To summarize, we have proved that $[(y_n)_{n \in \mathbb{N}}] \in (V \cap F_B) \setminus F_A$.

Finally, the set $X := \bigcup \{F_A : A \in \mathcal{I}\}$ is a closed subspace of E (by (2)) that can be covered by a family of nowhere dense subsets $\{F_A : A \in \mathcal{I}\} \subset \text{NWD}(X)$ (by (3)) with the property that $F_A \subset F_B$ whenever $A \subset B \in \mathcal{I}$ (by (1)). \square

Following [3], the *orthogonal* of an ideal \mathcal{I} on \mathbb{N} is defined as the set of all $A \in \mathcal{P}(\mathbb{N})$ such that $A \cap B \in \text{Fin}$ for every $B \in \mathcal{I}$. The ideal \mathcal{I} is said to be *tall* if its orthogonal is equal to Fin , or equivalently, if given $A \notin \text{Fin}$ there is $B \in \mathcal{I} \setminus \text{Fin}$ with $B \subset A$. We now extend these definitions.

Definition 4.5. Let $\mathcal{I}, \mathcal{I}_0$ be ideals on \mathbb{N} . The orthogonal of \mathcal{I} respect to \mathcal{I}_0 is defined as $\mathcal{I}^{\perp \mathcal{I}_0} := \{A \subset \mathbb{N} : A \cap B \in \mathcal{I}_0 \text{ for all } B \in \mathcal{I}\}$. We say that \mathcal{I} is \mathcal{I}_0 -tall if $\mathcal{I}^{\perp \mathcal{I}_0} = \mathcal{I}_0$, i.e. if for every $A \notin \mathcal{I}_0$ there exists $B \in \mathcal{I} \setminus \mathcal{I}_0$ such that $B \subset A$.

Proposition 4.6. Suppose that \mathcal{I} is a $\text{P}(\mathcal{I}_0)$ -ideal which is \mathcal{I}_0 -tall. Then every ideal \mathcal{J} containing $\mathcal{I} \cup \mathcal{I}_0$ is not a Baire ideal.

Proof. Following the notation of the proof of Proposition 4.4 consider the family $\{F_C : C \in \mathcal{I}\}$ of closed subsets of $E = \ell^\infty / c_0(\mathcal{I}_0)$ where $F_C = \{[(x_n)_{n \in \mathbb{N}}] : 0 \leq (x_n)_{n \in \mathbb{N}} \leq \chi_C\}$. We define a function $f : \mathcal{J} \rightarrow \mathcal{P}(E)$ that assigns to each $A \in \mathcal{J}$ the element

$$f(A) = \bigcup \{F_C : C \in \mathcal{I}, C \subset A\}.$$

Since $\bigcup \{f(A) : A \in \mathcal{J}\} = \bigcup \{F_C : C \in \mathcal{I}\} = X$ was a complete metric space, we just have to check that $f(A) \in \text{NWD}(X)$ for every $A \in \mathcal{J}$ to finish the proof. Fix $A \in \mathcal{J}$. Since $\mathbb{N} \setminus A \notin \mathcal{J} \supset \mathcal{I}_0$, there exists by hypothesis $B \in \mathcal{I} \setminus \mathcal{I}_0$ with $B \subset \mathbb{N} \setminus A$. Now we can repeat the argument of (3) in the proof of Proposition 4.4 to deduce that $f(A)$ is a nowhere dense subset of $f(A \cup B)$, and hence of X . \square

Corollary 4.7. If \mathcal{I} is a tall P -ideal on \mathbb{N} then every ideal $\mathcal{J} \supseteq \mathcal{I}$ is not a Baire ideal.

4.3. Ideals generated by AD families. Recall that a family of sets \mathcal{A} is said to be *almost disjoint* (AD) if the intersection of any pair of its elements is finite. We give now examples of ideals which do not belong to the class of $\mathcal{P}(\mathcal{I}_0)$ -ideals for any ideal \mathcal{I}_0 on \mathbb{N} .

Proposition 4.8. *Let \mathcal{I} be an ideal on \mathbb{N} generated by an infinite family \mathcal{A} of almost disjoint infinite sets. Then \mathcal{I} is not a $\mathcal{P}(\mathcal{I}_0)$ -ideal for any ideal \mathcal{I}_0 on \mathbb{N} .*

Proof. Suppose that \mathcal{I} is a $\mathcal{P}(\mathcal{I}_0)$ -ideal for some ideal \mathcal{I}_0 on \mathbb{N} . Fix $A \in \mathcal{A}$ and take a sequence $(B_n)_{n \in \mathbb{N}}$ in $\mathcal{A} \setminus \{A\}$ whose elements are all different. Now define inductively $A_0 := A$ and $A_{n+1} := A_n \cup B_n$ for every $n \in \mathbb{N}$. The assumption implies the existence of some $C = C_1 \cup \dots \cup C_k$ ($C_i \in \mathcal{A}$ for every i) such that $A_n \setminus C \in \mathcal{I}_0$ for every $n \geq 0$. But there must be some A_{n_0} different from the elements C_i 's. Therefore

$$A_{n_0} \setminus C = A_{n_0} \setminus ((A_{n_0} \cap C_1) \cup \dots \cup (A_{n_0} \cap C_k)) \in \mathcal{I}_0$$

and hence $A_{n_0} \in \mathcal{I}_0$ since $A_{n_0} \cap C_i$ is finite for each $i = 1, \dots, k$. But this means that $\mathcal{A} \subset \mathcal{I}_0$ and hence $\mathcal{I} \subset \mathcal{I}_0$, contradicting Definition 4.2 (II). \square

By a result of Mathias [12], ideals \mathcal{I} generated by a maximal almost disjoint (MAD) family of subsets of \mathbb{N} are not analytic, so they do not belong to any of the classes of ideals considered in the previous subsections. However, ideals generated by an AD family \mathcal{A} are not Baire ideals when the generating family \mathcal{A} has cardinality \mathfrak{c} , as it follows from the next result.

Proposition 4.9. *Let \mathcal{I} be an ideal on \mathbb{N} . Suppose that there exists a family $\mathcal{A} \subseteq \mathcal{I}$ with $|\mathcal{A}| = \mathfrak{c}$ and such that every $B \in \mathcal{I}$ satisfies $|\{A \in \mathcal{A} : A \subseteq B\}| < +\infty$. Then \mathcal{I} is not a Baire ideal for $2^{\mathbb{N}}$.*

Proof. We will use Corollary 3.2. Since \mathcal{A} has cardinality \mathfrak{c} we can index its elements as $\mathcal{A} = \{A_\sigma : \sigma \in 2^{\mathbb{N}}\}$ and define a function $f : 2^{<\mathbb{N}} \rightarrow \mathcal{P}(\mathcal{I})$ as

$$f(s) = \{B \in \mathcal{I} : A_\sigma \not\subseteq B \text{ whenever } s \preceq \sigma \in 2^{\mathbb{N}}\} \quad \text{for } s \in 2^{<\mathbb{N}}.$$

It is clearly monotonic. Fix any $s \in 2^{<\mathbb{N}}$. Obviously $f(s)$ is hereditary. Given $B \in \mathcal{I}$, since $\{\sigma \in 2^{\mathbb{N}} : A_\sigma \subset B\}$ is finite, we can find $t \succ s$ such that $A_\sigma \not\subseteq B$ for every $\sigma \succ t$, so that $B \in f(t)$. This shows that $\bigcup_{t \succ s} f(t) = \mathcal{I}$. Finally, given any $\sigma \in 2^{\mathbb{N}}$ we have that $A_\sigma \notin f(\sigma|_k)$ for every $k \in \mathbb{N}$. \square

5. AN UNCOUNTABLY GENERATED BAIRE IDEAL FOR EVERY POLISH SPACE

The aim of this section is to point out that there are models of ZFC in which we can find \mathcal{P} -ideals \mathcal{I} with character equal to \mathfrak{c} which are Baire ideals for every Polish space, despite of the fact that they are not Baire ideals for some (non-separable) complete metric space X_0 (see Proposition 4.4).

Let $M[G]$ be a model of set theory obtained by adding \aleph_2 many Cohen reals $\{x_\alpha : \alpha < \omega_2\} \subset 2^{\mathbb{N}}$ to a ground model M where Martin's Axiom and $\mathfrak{c} = \aleph_2$ hold. In the model M there exists an ω_2 -chain in $\mathcal{P}(\mathbb{N})/\text{Fin}$. That is, we have a family $\{C_\alpha\}_{\alpha < \omega_2}$ of subsets of \mathbb{N} such that $C_\alpha \subset^* C_\beta$ whenever $\alpha < \beta$; or

equivalently, $C_\alpha \setminus C_\beta$ is finite, but $C_\beta \setminus C_\alpha$ is infinite whenever $\alpha < \beta$. We define, in the model $M[G]$,

$$\mathcal{I} = \{a \subset \mathbb{N} : \exists \alpha : a \subset^* C_\alpha\}$$

It is clear that \mathcal{I} cannot be generated by less than \aleph_2 elements, and $\mathfrak{c} = \aleph_2$ in $M[G]$. We are going to prove that \mathcal{I} satisfies property (\square) for the case $X = 2^\mathbb{N}$. This in particular will imply that \mathcal{I} is a Baire ideal for $2^\mathbb{N}$, and by Corollary 3.2 this is equivalent to say that \mathcal{I} is a Baire ideal for every Polish space.

We will reason by contradiction assuming that we can write $2^\mathbb{N} = \bigcup_{A \in \mathcal{I}} X_A$ in such a way that each X_A is meager and $A \subset B$ implies $X_A \subset X_B$. If we consider $Y_A = \bigcup \{X_{A \cup F} : F \subset \mathbb{N} \text{ is finite}\}$, we still have that each Y_A is meager, $2^\mathbb{N} = \bigcup_{A \in \mathcal{I}} Y_A$ and now $A \subset^* B$ implies $Y_A \subset Y_B$. Given the definition of \mathcal{I} , this implies that $2^\mathbb{N} = \bigcup_{\alpha < \omega_2} Y_{C_\alpha}$ and $Y_{C_\alpha} \subset Y_{C_\beta}$ if $\alpha < \beta$. In particular, there must exist α such that $\{x_\gamma : \gamma < \omega_1\} \subset Y_{C_\alpha}$, and in particular, $\{x_\gamma : \gamma < \omega_1\}$ would be a meager set. However, this contradicts the following general fact about Cohen reals, whose proof we include for the reader's convenience:

Proposition 5.1. *If $Z \subset \omega_1$ is an uncountable set, then $\{x_\alpha : \alpha \in Z\}$ fails to be nowhere dense in $2^\mathbb{N}$.*

Proof. Let \mathbb{P} be the forcing that adds the Cohen reals $\{x_\alpha : \alpha < \omega_2\}$, and let $\{\dot{x}_\alpha : \alpha < \omega_2\}$ be the canonical names of these Cohen reals. Remember, elements $p \in \mathbb{P}$ are $\{0, 1\}$ -valued functions whose domain is a finite subset of $\mathbb{N} \times \omega_2$, endowed with the natural extension order that $p < q$ iff $\text{dom}(p) \supset \text{dom}(q)$ and $p|_{\text{dom}(q)} = q$. For each $s \in 2^{<\mathbb{N}}$ we can consider the clopen set $[s]$ of $2^\mathbb{N}$ consisting of all infinite sequence that end-extend s . This describes a basis of open subsets of $2^\mathbb{N}$. Moreover $[t] \subset [s]$ if and only if t is an end-extension of s , that we write as $s \prec t$. So, if $\{x_\alpha : \alpha \in Z\}$ is nowhere dense, then there exists a function $t : 2^{<\mathbb{N}} \rightarrow 2^{<\mathbb{N}}$ such that, for all $s \in 2^{<\mathbb{N}}$:

- (1) $s \prec t(s)$,
- (2) $[t(s)] \cap \{x_\alpha : \alpha \in Z\} = \emptyset$

Let \dot{t} be a name for t . For every $s \in 2^{<\mathbb{N}}$ there must exist a condition $p_s \in \mathbb{P}$ such that $p_s \Vdash \dot{t}(s) = t(s)$. Since $2^{<\mathbb{N}}$ is countable and each p_s has finite domain, we can find $\alpha \in Z$ such that $(\mathbb{N} \times \{\alpha\}) \cap \text{dom}(p_s) = \emptyset$ for all $s \in 2^{<\mathbb{N}}$. We know, by the definition of the function t , that $x_\alpha \notin [t(s)]$ for any $s \in 2^{<\mathbb{N}}$. However, we are going to prove that for every condition q in the generic filter of \mathbb{P} there exists $q' < q$ such that

$$q' \Vdash \exists s \in 2^{<\mathbb{N}} : \dot{x}_\alpha \in [\dot{t}(s)]$$

and this is a contradiction. So fix $q \in G$. We can assume that the domain of q is of the form $\{0, \dots, n_0\} \times F_q$ with $\alpha \in F_q$. Consider $s \in 2^{<\mathbb{N}}$ given by $s(n) = q(n, \alpha)$ for $n \leq n_0$. We define the desired condition q' as $q'(n, \gamma) = p_s(n, \gamma)$ if $(n, \gamma) \in \text{dom}(p_s)$, and $q'(n, \alpha) = t(s)(n)$ for any n where $t(s)$ is defined. This is possible by the choice we made of α . This ensures, on the one hand, that $q' \leq p_s$, so $q' \Vdash \dot{t}(s) = t(s)$. On the other hand, by the way q' is defined on $\mathbb{N} \times \{\alpha\}$, $q' \Vdash \dot{x}_\alpha \in [\dot{t}(s)]$. Thus $q' \Vdash \dot{x}_\alpha \in [\dot{t}(s)]$ as desired. \square

6. COVERINGS WITH COMPACT SUBSETS

In this section we present an example which shows that an analogue of Baire Theorem for coverings of $\mathbb{N}^{\mathbb{N}}$ by compact subsets can be valid for some uncountably generated analytic ideals. For an arbitrary topological space X , we write $\mathcal{K}(X) := \{A \subseteq X : A \text{ is compact}\}$. Given a partially ordered set (D, \leq) , we say that a subset $A \subseteq D$ is *weakly bounded* if each infinite subset of A contains an infinite bounded subset.

Proposition 6.1. *Let $f: \mathcal{I} \rightarrow \mathcal{K}(X)$ be a monotonic function where \mathcal{I} is an ideal on \mathbb{N} . If $\mathcal{B} \subset \mathcal{I}$ is a weakly bounded subset then $\bigcup \{f(B) : B \in \mathcal{B}\}$ is countably compact.*

Proof. Let $(x_n)_{n \in \mathbb{N}}$ be an infinite sequence in $\bigcup \{f(B) : B \in \mathcal{B}\}$. We can choose for each $n \in \mathbb{N}$ an element $B_n \in \mathcal{B}$ so that $x_n \in f(B_n)$. Either if the range of $(B_n)_{n \in \mathbb{N}}$ is finite or not, we can find by the hypothesis a subsequence $(B_{n_k})_{k \in \mathbb{N}}$ bounded by a set $A \in \mathcal{I}$. The corresponding subsequence $(b_{n_k})_{k \in \mathbb{N}}$ is contained in $f(A)$ by monotonicity, so it has a cluster point in $f(A)$ which will also be a cluster point of the original sequence $(b_n)_{n \in \mathbb{N}}$. \square

Corollary 6.2. *Let \mathcal{I} be an ideal on \mathbb{N} that can be written as a countable union of weakly bounded subsets. Then, every monotonic function $f: \mathcal{I} \rightarrow \mathcal{K}(\mathbb{N}^{\mathbb{N}})$ satisfy that $\bigcup \{f(A) : A \in \mathcal{I}\} \neq \mathbb{N}^{\mathbb{N}}$.*

Proof. Suppose that $\mathcal{I} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ where \mathcal{B}_n is a weakly bounded subset of \mathcal{I} . Then $\bigcup \{f(B) : B \in \mathcal{B}_n\}$ is a compact subset of $\mathbb{N}^{\mathbb{N}}$, and hence $\bigcup \{f(A) : A \in \mathcal{I}\}$ is a countable union of compact sets. But $\mathbb{N}^{\mathbb{N}}$ is not a countable union of compact sets as an easy application of Baire Category Theorem. \square

We show now an example, taken from [9, p. 178, Example 1], of an ideal that satisfies the hypothesis of Corollary 6.2. Consider the map $\varphi: \mathcal{P}(\mathbb{N}) \rightarrow [0, +\infty]$ defined by

$$\varphi(A) = \inf \{c > 0 : |A \cap \{1, \dots, 2^n\}| \leq n^c \text{ for each } n \geq 2\}$$

with the convention $\varphi(A) = +\infty$ if the infimum does not exist. This is a lower-semicontinuous submeasure, so the ideal $\mathcal{I}_\varphi = \{A \subset \mathbb{N} : \varphi(A) < +\infty\}$ is an F_σ ideal (see [3, p. 7, Lemma 1.2.2]). It is easy to check that \mathcal{I} is not countably generated, and it is a countable union of weakly bounded subsets since for each $c > 0$

$$\mathcal{I}_\varphi(c) = \{A \subset \mathbb{N} : \varphi(A) < c\}$$

is weakly bounded, see [9, p. 178, Example 1] for the details.

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